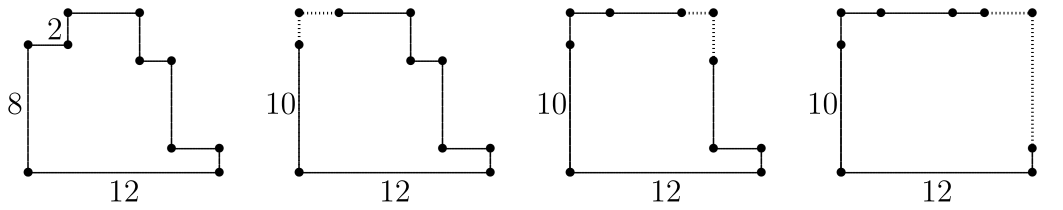
Note: Throughout this pamphlet, $[P_1P_2\ldots P_n]$ denotes the area enclosed by polygon $P_1P_2\ldots P_n$.

1. **(C)** Since $\mathtt{a}\times 3$ has units digit 9, $\mathtt{a}$ must be 3. Hence $\mathtt{b}\times 3$ has units digit 2, so $\mathtt{b}$ must be 4. Thus, $\mathtt{a}+\mathtt{b}=7$.

2. **(D)** Each polygon in the sequence below has the same perimeter, which is 44.



3. **(D)** Since each summand is nonnegative, the sum is zero only when each term is zero. Hence the only solution is $x=3,\ y=4,$ and $z=5$, so the desired sum is 12.

4. **(A)** If a is 50% larger than c, then $a=1.5c$. If b is 25% larger than c, then $b=1.25c$. So $\frac{a}{b}=\frac{1.5c}{1.25c}=\frac{6}{5}=1.20$, and $a=1.20b$. Therefore, a is 20% larger than b.

5. **(C)** Let x and y denote the width and height of one of the five rectangles, with $x<y$. Then $5x+4y=176$ and $3x=2y$. Solve simultaneously to get $x=16$ and $y=24$. The perimeter in question is $2{\cdot}16+2{\cdot}24=80$.

6. **(B)** The 200 terms can be grouped into 100 odd-even pairs, each with a sum of -1. Thus the sum of the first 200 terms is $-1\cdot 100=-100$, and the average of the first 200 terms is $-100/200=-0.5$.

7. **(D)** Not all seven integers can be larger than 13. If six of them were each 14, then the seventh could be $-(6\times 14)-1$, so that the sum would be -1.

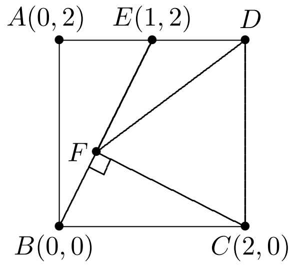
8. **(D)** The cost of 25 books is $C(25)=25\times\$11=\$275$. The cost of 24 books is $C(24)=24\times\$12=\$288$, while 23 and 22 books cost $C(23)=23\times\$12=\$276$ and $C(22)=22\times\$12=\$264$, respectively. Thus it is cheaper to buy 25 books than 23 or 24 books. Similarly, 49 books cost less than 45, 46, 47, or 48 books. In these six cases the total cost is reduced by ordering more books. There are no other cases.

OR

A discount of \$1 per book is given on orders of at least 25 books. This discount is larger than $2 \times \$12$, the cost of two books at the regular price. Thus, $n = 23$ and $n = 24$ are two values of n for which it is cheaper to order more books. Similarly, we receive an additional \$1 discount per book when we buy at least 49 books. This discount would enable us to buy 4 more books at \$10 per book, so there are 4 more values of n: $45, 46, 47,$ and 48, for a total of 6 values.

9. **(C)** In right triangle BAE, $BE = \sqrt{2^2 + 1^2} = \sqrt{5}$. Since $\triangle CFB \sim \triangle BAE$, it follows that $[CFB] = \left(\frac{CB}{BE}\right)^2 \cdot [BAE] = \left(\frac{2}{\sqrt{5}}\right)^2 \cdot \frac{1}{2}(2 \cdot 1) = \frac{4}{5}$. Then $[CDEF] = [ABCD] - [BAE] - [CFB] = 4 - 1 - \frac{4}{5} = \frac{11}{5}$.

OR



Draw the figure in the plane as shown with B at the origin. An equation of the line BE is $y = 2x$, and, since the lines are perpendicular, an equation of the line CF is $y = -\frac{1}{2}(x - 2)$. Solve these two equations simultaneously to get $F = (2/5, 4/5)$ and

$$[CDEF] = [DEF] + [CDF] = \frac{1}{2}(1)\left(2 - \frac{4}{5}\right) + \frac{1}{2}(2)\left(2 - \frac{2}{5}\right) = \frac{11}{5}.$$

10. **(D)** There are 36 equally likely outcomes as shown in the following table.

(1, 1)	**(1,2)**	**(1,4)**	**(1,4)**	(1, 5)	**(1,6)**
(2,1)	(2, 2)	(2, 4)	(2, 4)	**(2,5)**	(2, 6)
(3, 1)	**(3,2)**	**(3,4)**	**(3,4)**	(3, 5)	**(3,6)**
(3, 1)	**(3,2)**	**(3,4)**	**(3,4)**	(3, 5)	**(3,6)**
(5, 1)	**(5,2)**	**(5,4)**	**(5,4)**	(5, 5)	**(5,6)**
(6,1)	(6, 2)	(6, 4)	(6, 4)	**(6,5)**	(6, 6)

Exactly 20 of the outcomes have an odd sum. Therefore, the probability is $\frac{20}{36} = \frac{5}{9}$.

OR

The sum is odd if and only if one number is even and the other is odd. The probability that the first number is even and the second is odd is $\frac{1}{3} \cdot \frac{1}{3}$, and the probability that the first is odd and the second is even is $\frac{2}{3} \cdot \frac{2}{3}$. Therefore, the required probability is $(\frac{1}{3})^2 + (\frac{2}{3})^2 = \frac{5}{9}$.

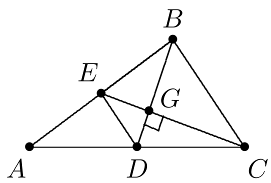
11. **(D)** The average for games six through nine was $(23+14+11+20)/4 = 17$, which exceeded her average for the first five games. Therefore, she scored at most $5 \cdot 17 - 1 = 84$ points in the first five games. Because her average after ten games was more than 18, she scored at least 181 points in the ten games, implying that she scored at least $181-84-68 = 29$ points in the tenth game.

12. **(E)** Since $mb > 0$, the slope and the y-intercept of the line are either both positive or both negative. In either case, the line slopes away from the positive x-axis and does not intersect it. The answer is therefore $(1997, 0)$. Note that the other four points lie on lines for which $mb > 0$. For example, $(0, 1997)$ lies on $y = x + 1997$; $(0, -1997)$ lies on $y = -x - 1997$; $(19, 97)$ lies on $y = 5x + 2$; and $(19, -97)$ lies on $y = -5x - 2$.

13. **(E)** Let $N = 10x + y$. Then $10x + y + 10y + x = 11(x + y)$ must be a perfect square. Since $1 \le x + y \le 18$, it follows that $x + y = 11$. There are eight such numbers: 29, 38, 47, 56, 65, 74, 83, and 92.

14. **(B)** Let x be the number of geese in 1996, and let k be the constant of proportionality. Then $x - 39 = 60k$ and $123 - 60 = kx$. Solve the second equation for k, and use that value to solve for x in the first equation, obtaining $x - 39 = 60 \cdot \frac{63}{x}$. Thus $x^2 - 39x - 3780 = 0$. Factoring yields $(x-84)(x+45) = 0$. Since x is positive, it follows that $x = 84$.

15. **(D)** Let the medians meet at G. Then $CG = (2/3)CE = 8$ and the area of triangle BCD is $(1/2)BD \cdot CG = (1/2) \cdot 8 \cdot 8 = 32$. Since BD is a median, triangles ABD and DBC have the same area. Hence the area of the triangle is 64.



OR

Since the medians are perpendicular, the area of the quadrilateral $BCDE$ is half the product of the diagonals $\frac{1}{2}(12)(8) = 48$. (Why?) However, D and E are midpoints, which makes the area of triangle AED one fourth of the area of triangle ABC. Thus the area of $BCDE$ is three fourths of the area of triangle ABC. It follows that the area of triangle ABC is 64.

16. **(D)** If only three entries are altered, then either two lines are not changed at all, or some entry is the only entry in its row and the only entry in its column that is changed. In either case, at least two of the six sums remain the same. However, four alterations are enough. For example, replacing 4 by 5, 1 by 3, 2 by 7, and 6 by 9 results in the array

$$\begin{bmatrix} 5 & 9 & 7 \\ 8 & 3 & 9 \\ 3 & 5 & 7 \end{bmatrix}$$

for which the six sums are all different.

17. **(A)** The line $x = k$ intersects $y = \log_5(x+4)$ and $y = \log_5 x$ at $(k, \log_5(k+4))$ and $(k, \log_5 k)$, respectively. Since the length of the vertical segment is 0.5,

$$0.5 = \log_5(k+4) - \log_5 k = \log_5 \frac{k+4}{k},$$

so $\dfrac{k+4}{k} = \sqrt{5}$. Solving for k yields $k = \dfrac{4}{\sqrt{5}-1} = 1 + \sqrt{5}$, so $a + b = 6$.

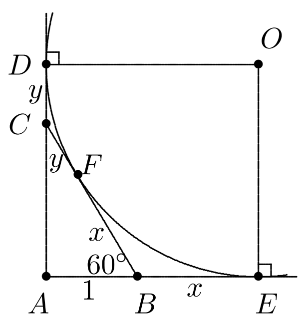
18. **(E)** When 10 is added to a number in the list, the mean increases by 2, so there must be five numbers in the original list whose sum is $5 \cdot 22 = 110$. Since 10 is the smallest number in the list and m is the median, we may assume

$$10 \le a \le m \le b \le c,$$

denoting the other members of the list by a, b, and c. Since the mode is 32, we must have $b = c = 32$; otherwise, $10 + m + a + b + c$ would be larger than 110. So $a + m = 36$. Since decreasing m by 8 decreases the median by 4, a must be 4 less than m. Solving $a + m = 36$ and $m - a = 4$ for m gives $m = 20$.

19. **(D)** Let D and E denote the points of tangency on the y- and x-axes, respectively, and let $\overline{BC}$ be tangent to the circle at F. Tangents to a circle from a point are equal, so $BE = BF$ and $CD = CF$. Let $x = BF$ and $y = CF$. Because $x + y = BC = 2$, the radius of the circle is

$$\frac{(1+x) + (\sqrt{3}+y)}{2} = \frac{3+\sqrt{3}}{2} \approx 2.37.$$



OR

Let r be the radius of the circle. The area of square $AEOD$, r^2, may also be expressed as the sum of the areas of quadrilaterals $OFBE$ and $ODCF$ and triangle ABC. This is given by $rx + ry + \frac{\sqrt{3}}{2}$, where $x + y = 2$. Thus

$$r^2 = 2r + \frac{\sqrt{3}}{2}.$$

Solving for r using the quadratic formula yields the positive solution

$$r = 1 + \sqrt{1 + \frac{\sqrt{3}}{2}} \approx 2.37.$$

Note. The circle in question is called an *escribed* circle of the triangle ABC.

20. **(A)** Since

$$1 + 2 + 3 + \cdots + 100 = (100)(101)/2 = 5050,$$

it follows that the sum of any sequence of 100 consecutive positive integers starting with $a + 1$ is of the form

$$\begin{aligned}(a+1) + (a+2) + (a+3) + \cdots + (a+100) &= 100a + (1 + 2 + 3 + \cdots + 100) \\ &= 100a + 5050.\end{aligned}$$

Consequently, such a sum has 50 as its rightmost two digits. Choice **A** is the sum of the 100 integers beginning with 16,273,800.

21. **(C)** Since $\log_8 n = \frac{1}{3}(\log_2 n)$, it follows that $\log_8 n$ is rational if and only if $\log_2 n$ is rational. The nonzero numbers in the sum will therefore be all numbers of the form $\log_8 n$, where n is an integral power of 2. The highest power of 2 that does not exceed 1997 is 2^{10}, so the sum is:

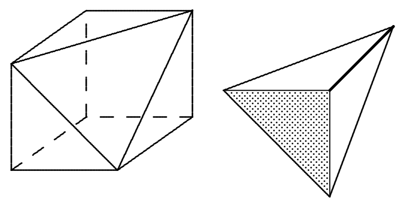
$$\log_8 1 + \log_8 2 + \log_8 2^2 + \log_8 2^3 + \cdots + \log_8 2^{10} =$$

$$0 + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \cdots + \frac{10}{3} = \frac{55}{3}.$$

Challenge. Prove that $\log_2 3$ is irrational. Prove that, for every integer n, $\log_2 n$ is rational if and only if n is an integral power of 2.

22. **(E)** Let A, B, C, D, and E denote the amounts Ashley, Betty, Carlos, Dick, and Elgin had for shopping, respectively. Then $A - B = \pm 19$, $B - C = \pm 7$, $C - D = \pm 5$, $D - E = \pm 4$, and $E - A = \pm 11$. The sum of the left sides is zero, so the sum of the right sides must also be zero. In other words, we must choose some subset S of $\{4, 5, 7, 11, 19\}$ which has the same element-sum as its complement. Since $4 + 5 + 7 + 11 + 19 = 46$, the sum of the members of S is 23. Hence S is either the set $\{4, 19\}$ or its complement $\{5, 7, 11\}$. Thus either $A - B$ and $D - E$ are the only positive differences or $B - C$, $C - D$, and $E - A$ are. In the former case, expressing A, B, C, and D in terms of E, we get $5E + 6 = 56$, which yields $E = 10$. In the latter case, the same strategy yields $5E - 6 = 56$, which leads to non-integer values. Hence $E = 10$.

23. **(D)** The polyhedron is a unit cube with a corner cut off. The missing corner may be viewed as a pyramid whose altitude is 1 and whose base is an isosceles right triangle (shaded in the figure). The area of the base is $1/2$. The pyramid's volume is therefore $(1/3)(1/2)(1) = 1/6$, so the polyhedron's volume is $1 - 1/6 = 5/6$.



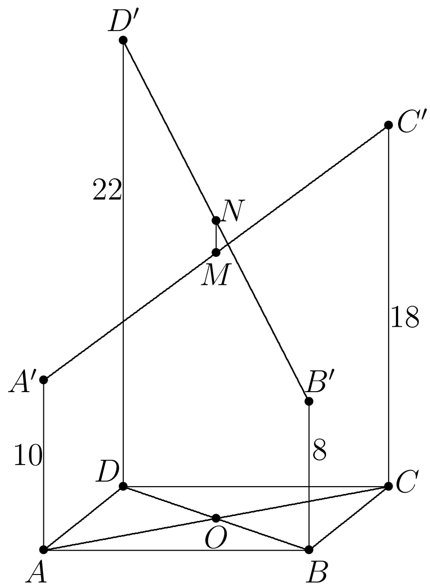
24. **(B)** The number of five-digit rising numbers that begin with 1 is $\binom{8}{4} = 70$, since the rightmost four digits must be chosen from the eight-member set $\{2, 3, 4, 5, 6, 7, 8, 9\}$, and, once they are chosen, they can be arranged in increasing order in just one way. Similarly, the next $\binom{7}{4} = 35$ integers in the list begin with 2. So the 97^{th} integer in the list is the 27^{th} among those that begin with 2. Among those that begin with 2, there are $\binom{6}{3} = 20$ that begin with 23 and $\binom{5}{3} = 10$ that begin with 24. Therefore, the 97^{th} is the 7^{th} of those that begin with 24. The first six of those beginning with 24 are $24567, 24568, 24569, 24578, 24579, 24589$, and the seventh is 24678. The digit 5 is not used in the representation.

OR

As above, note that there are 105 integers in the list starting with either 1 or 2, so the 97^{th} one is ninth from the end. Count backwards: $26789, 25789, 25689, 25679, 25678, 24789, 24689, 24679$, $\boxed{24678}$. Thus 5 is a missing digit.

25. **(B)** Let O be the intersection of $\overline{AC}$ and $\overline{BD}$. Then O is the midpoint of $\overline{AC}$ and $\overline{BD}$, so $\overline{OM}$ and $\overline{ON}$ are the midlines in trapezoids $ACC'A'$ and $BDD'B'$, respectively. Hence $OM = (10+18)/2 = 14$ and $ON = (8+22)/2 = 15$. Since $OM \| AA'$, $ON \| BB'$, and $AA' \| BB'$, it follows that O, M, and N are collinear. Therefore,

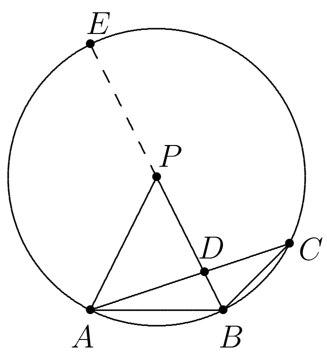
$$MN = |OM - ON| = |14 - 15| = 1.$$



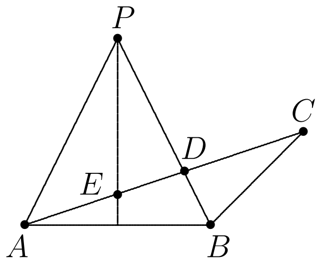
Note. In general, if $AA' = a$, $BB' = b$, $CC' = c$, and $DD' = d$, then $MN = |a - b + c - d|/2$.

26. **(A)** Construct a circle with center P and radius PA. Then C lies on the circle, since the angle ACB is half angle APB. Extend $\overline{BP}$ through P to get a diameter $\overline{BE}$. Since A, B, C, and E are concyclic,

$$\begin{aligned} AD \cdot CD &= ED \cdot BD \\ &= (PE + PD)(PB - PD) \\ &= (3+2)(3-2) \\ &= 5. \end{aligned}$$



OR



Let E denote the point where $\overline{AC}$ intersects the angle bisector of angle APB. Note that $\triangle PED \sim \triangle CBD$. Hence $DE/2 = 1/DC$ so $DE \cdot DC = 2$. Apply the *Angle Bisector Theorem* to $\triangle APD$ to obtain

$$\frac{EA}{DE} = \frac{PA}{PD} = \frac{3}{2}.$$

Thus $DA \cdot DC = (DE + EA) \cdot DC = (DE + 1.5\,DE) \cdot DC = 2.5\,DE \cdot DC = 5.$

27. **(D)** We may replace x with $x + 4$ in

$$f(x+4) + f(x-4) \;=\; f(x) \qquad (1)$$

to get

$$f(x+8) + f(x) \;=\; f(x+4). \qquad (2)$$

From (1) and (2), we deduce that $f(x+8) = -f(x-4)$. Replacing x with $x+4$, the latter equation yields $f(x+12) = -f(x)$. Now replacing x in this last equation with $x+12$ yields $f(x+24) = -f(x+12)$. Consequently, $f(x+24) = f(x)$ for all x, so that a least period p exists and is at most 24. On the other hand, the function $f(x) = \sin\left(\frac{\pi x}{12}\right)$ has fundamental period 24, and satisfies (1), so $p \geq 24$. Hence $p = 24$.

OR

Let x_0 be arbitrary, and let $y_k = f(x_0+4k)$ for $k = 0, 1, 2, \ldots$. Then $f(x+4) = f(x) - f(x-4)$ for all x implies $y_{k+1} = y_k - y_{k-1}$, so if $y_0 = a$ and $y_1 = b$, then $y_2 = b - a$, $y_3 = -a$, $y_4 = -b$, $y_5 = a - b$, $y_6 = a$, and $y_7 = b$. It follows that the sequence (y_k) is periodic with period 6 and, since x_0 was arbitrary, f is periodic with period 24. Since $f(x) = \sin(\frac{\pi x}{12})$ has fundamental period 24 and satisfies $f(x+4) + f(x-4) = f(x)$, it follows that $p \geq 24$. Hence $p = 24$.

28. **(E)** If $c \geq 0$, then $ab - |a+b| = 78$, so $(a-1)(b-1) = 79$ or $(a+1)(b+1) = 79$. Since 79 is prime, $\{a, b\}$ is $\{2, 80\}$, $\{-78, 0\}$, $\{0, 78\}$, or $\{-80, -2\}$. Hence $|a + b| = 78$ or $|a + b| = 82$, and, from the first equation in the problem statement, it follows that $c < 0$, a contradiction.

On the other hand, if $c < 0$, then $ab + |a + b| = 116$, so $(a + 1)(b + 1) = 117$ in the case that $a + b > 0$ and $(a - 1)(b - 1) = 117$ in the case that $a + b < 0$. Since $117 = 3^2 \cdot 13$, we distinguish the following cases:

$$\begin{aligned} \{a,b\} &= \{0,116\} &\quad \text{yields} \quad & c = -97; \\ \{a,b\} &= \{2,38\} &\quad \text{yields} \quad & c = -21; \\ \{a,b\} &= \{8,12\} &\quad \text{yields} \quad & c = -1; \\ \{a,b\} &= \{-116,0\} &\quad \text{yields} \quad & c = -97; \\ \{a,b\} &= \{-38,-2\} &\quad \text{yields} \quad & c = -21; \\ \{a,b\} &= \{-12,-8\} &\quad \text{yields} \quad & c = -1. \end{aligned}$$

Since a and b are interchangeable, each of these cases leads to two solutions, for a total of 12.

29. **(B)** Suppose $1 = x_1 + x_2 + \cdots + x_n$ where $x_1, x_2, \ldots, x_n$ are special and $n \leq 9$. For $k = 1, 2, 3, \ldots$, let a_k be the number of elements of $\{x_1, x_2, \ldots, x_n\}$ whose k^{th} decimal digit is 7. Then

$$1 = \frac{7a_1}{10} + \frac{7a_2}{10^2} + \frac{7a_3}{10^3} + \cdots,$$

which yields

$$\frac{1}{7} = 0.\overline{142857} = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots.$$

Hence $a_1 = 1, a_2 = 4, a_3 = 2, a_4 = 8$, etc. In particular, this implies that $n \geq 8$. On the other hand,

$x_1 = 0.\overline{700}$, $x_2 = x_3 = 0.\overline{07}$, $x_4 = x_5 = 0.\overline{077777}$, and $x_6 = x_7 = x_8 = 0.\overline{000777}$

are 8 special numbers whose sum is

$$\frac{700700 + 2(70707) + 2(77777) + 3(777)}{999999} = 1.$$

Thus the smallest n is 8.

30. **(C)** In order that $D(n) = 2$, the binary representation of n must consist of a block of 1's followed by a block of 0's followed by a block of 1's. Among the integers n with d-digit binary representations, how many are there for which $D(n) = 2$? If the 0's block consists of just one 0, there are $d - 2$ possible locations for the 0. If the block consists of multiple 0's, then there are $\binom{d-2}{2}$ such blocks, since only the first and last places for the 0's need to be identified. Thus there are $(d-2) + \frac{1}{2}(d-2)(d-3) = \frac{1}{2}(d-2)(d-1)$ values of n with d binary digits such that $D(n) = 2$. The binary representation of 97 has seven digits, so all the 3-, 4-, 5-, and 6-digit binary integers are less than 97. (We need not consider the 1- and 2-digit binary integers.) The sum of the values of $\frac{1}{2}(d-2)(d-1)$ for $d = 3, 4, 5,$ and 6 is 20. We must also consider the 7-digit binary integers less than or equal to $1100001_2 = 97$. If the initial block of 1's contains three or more 1's, then the number would be greater than 97; by inspection, if there are one or two 1's in the initial 1's block, there are respectively five or one acceptable configurations of the 0's block. It follows that the number of solutions of $D(n) = 2$ within the required range is $20 + 5 + 1 = 26$.

OR

Note that $D(n) = 2$ holds exactly when the binary representation of n consists of an initial block of 1's, followed by a block of 0's, and then a final block of 1's. The number of nonnegative integers $n \le 2^7 - 1 = 127$ for which $D(n) = 2$ is thus $\binom{7}{3} = 35$, since for each n, the corresponding binary representation is given by selecting the position of the leftmost bit in each of the three blocks. If $98 \le n \le 127$, the binary representation of n is either (a) $110XXXX_2$ or (b) $111XXXX_2$. Consider those n's for which $D(n) = 2$. By the same argument as above, there are three of type (a), namely $1101111_2 = 111$, $1100111_2 = 103$, and $1100011_2 = 99$. There are $\binom{4}{2} = 6$ of type (b). It follows that the number of solutions of $D(n) = 2$ for which $1 \le n \le 97$ is $35 - (3 + 6) = 26$.